

(Non)triviality of Pure Spinors and Exact Pure Spinor - RNS Map

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Abstract

All the BRST-invariant operators in pure spinor formalism in $d = 10$ can be represented as BRST commutators, such as $V = \{Q_{brst}, \frac{\theta_+}{\lambda_+} V\}$ where λ_+ is the $U(5)$ component of the pure spinor transforming as $1_{\frac{5}{2}}$. Therefore, in order to secure non-triviality of BRST cohomology in pure spinor string theory, one has to introduce “small Hilbert space” and “small operator algebra” for pure spinors, analogous to those existing in RNS formalism. As any invariant vertex operator in RNS string theory can also be represented as a commutator $V = \{Q_{brst}, LV\}$ where $L = -4c\partial\xi\xi e^{-2\phi}$, we show that mapping $\frac{\theta_+}{\lambda_+}$ to L leads to identification of the pure spinor variable λ^α in terms of RNS variables without any additional non-minimal fields. We construct the RNS operator satisfying all the properties of λ^α and show that the pure spinor BRST operator $\oint \lambda^\alpha d_\alpha$ is mapped (up to similarity transformation) to the BRST operator of RNS theory under such a construction.

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Introduction

Pure spinor formalism for superstrings has been proposed by Berkovits several years ago [1] as an alternative method of covariant quantization of Green-Schwarz superstring theory [2]. It involves the remarkably simple worldsheet action:

$$S = \int d^2z \left\{ \frac{1}{2} \partial X_m \bar{\partial} X^m + p_\alpha \bar{\partial} \theta^\alpha + \bar{p}_\alpha \partial \bar{\theta}^\alpha + \lambda_\alpha \bar{\partial} w^\alpha + \bar{\lambda}_\alpha \partial \bar{w}^\alpha \right\} \quad (1)$$

where p_α is conjugate to θ_α [3] and the commuting spinors λ^α and w^α are the bosonic ghosts which, roughly speaking, are related to the fermionic gauge κ -symmetry in GS superstring theory. The action (1) is related to the standard GS action by substituting the constraint

$$d_\alpha = p_\alpha - \frac{1}{2} (\partial X^m + \frac{1}{4} \theta \gamma^m \partial \theta) (\gamma^m \theta)_\alpha = 0 \quad (2)$$

and the corresponding BRST operator

$$Q_{brst} = \oint \frac{dz}{2i\pi} \lambda^\alpha d_\alpha(z) \quad (3)$$

is nilpotent provided that λ^α satisfies the pure spinor condition:

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0 \quad (4)$$

reducing the number of independent components of λ from 16 to 11. An example of unintegrated massless vertex operator in such a BRST cohomology is given by

$$U = \lambda^\alpha A_\alpha(X, \theta) \quad (5)$$

This operator is physical provided that the space-time superfield A_α is on-shell:

$$\gamma_{m_1 \dots m_5}^{\alpha\beta} D_\alpha A_\beta = 0 \quad (6)$$

(this particularly implies the standard Maxwell equation for the bosonic vector component of A) and thus the vertex operator (5) is identified with the emission of a photon by the superstring [1], [4], [5]. The integrated version of this operator $\sim \oint \frac{dz}{2i\pi} V(z)$ satisfying $[Q_{brst}, V] = \partial U$ can also be constructed, with V obviously having ghost number zero [6]. Physical vertex operators (both massless and massive) considered in pure spinor formalism

thus typically have ghost number 1 in unintegrated form and number zero in the integrated version.

The important question is how the PS approach is related to other descriptions of superstring, such as RNS formalism. While such a relation exists and can be constructed, the construction is not straightforward and the constructions considered so far particularly required the introduction of additional non-minimal fields by hands [7], [6]

Another natural question is whether the PS superstring could contain any additional physical operators, e.g. with higher ghost numbers. It is far from obvious that such operators could exist at all. For example, a straightforward naive attempt to generalize the unintegrated operator (5) to the ghost number 2 case:

$$U_2 = \lambda^\alpha \lambda^\beta F_{\alpha\beta}(X, \theta) \quad (7)$$

fails as the on-shell conditions for the field $F_{\alpha\beta}$:

$$\gamma_{m_1 \dots m_5}^{\alpha\beta} D_\gamma F_{\alpha\beta}(X, \theta) = 0 \quad (8)$$

imply the triviality of the U_2 operator:

$$U_2 = \{Q_{brst}, \theta^\alpha \lambda^\beta F_{\alpha\beta}\} \quad (9)$$

Similarly, naive construction of ghost number n operators $\sim \lambda^n$ leads to BRST-exact expressions, provided the on-shell constraint on the corresponding background fields. Despite that, below we shall demonstrate that vertex operators with non-standard coupling to pure spinors do appear in BRST cohomology. In general, the question of non-triviality of BRST cohomology in the PS formalism appears more subtle compared to RNS. That is, since $\{Q, \theta^\alpha\} = \lambda^\alpha$ and $[Q, \lambda^\alpha] = 0$, any invariant operator V in pure spinor string theory can be written as an exact BRST commutator. For example, consider the standard $U(5)$ -invariant parametrization of λ^α : $\lambda^\alpha = (\lambda^+, \lambda^{ab}, \lambda^a)$ ($a, b = 1, \dots, 5$) with $\lambda_{ab} = -\lambda_{ba}$ and $\lambda^a = \epsilon^{abcde} \lambda_{bc} \lambda_{de}$. Then any invariant V can be written as

$$V = [Q_{brst}, \frac{\theta_+}{\lambda^+} V] \quad (10)$$

This poses a question whether BRST cohomology of PS string theory is empty (similar observations have also been made in [8])

In fact, the identity (10) is reminiscent of the similar relation in the RNS formalism where any invariant V can be written as

$$V = \{Q_{brst}^{RNS}, LV\} \quad (11)$$

where

$$L = -4ce^{2\chi-2\phi} = -4c\partial\xi\xi e^{-2\phi} \quad (12)$$

with the ghost fields bosonized as [9]

$$\begin{aligned} b &= e^{-\sigma}, c = e^\sigma \\ \beta &= e^{\chi-\phi}\partial\chi \equiv \partial\xi e^{-\phi}, \gamma = e^{\phi-\chi} \end{aligned} \quad (13)$$

It is easy to check

$$\{Q_{brst}^{RNS}, L\} = 1 \quad (14)$$

In RNS approach, however, the relation (12) does not lead to the triviality of states since the L -operator is not in the small operator algebra, as it explicitly depends on $\xi = e^\chi$ (rather than its derivatives). So the only way to bail out pure spinors is to introduce similar classification for the PS formalism as well. Such a classification, however, isn't as obvious as in the RNS case. In the RNS case we exclude the operators with explicit ξ -dependence because the bosonization relations for the ghost fields β and γ depend on the derivative of ξ , but not on ξ itself (ξ can only be expressed as a generalized step function of β : $\xi = \Theta(\beta)$) In the PS formalism, however, the analogue of the L -operator is given by the ratio $\frac{\theta_+}{\lambda^+}$ consisting of fields already present in the theory. For this reason, the distinction between “large” and “small” operator algebras appears more obscure in the PS approach. One possible approach is to try to construct a direct map between PS and RNS variables, which in particular would identify $\frac{\theta_+}{\lambda^+}$ with the L -operator of RNS formalism. Once such a map is constructed, it would transform the states from the little Hilbert space in RNS formalism to those in the little Hilbert space in the pure spinor description. So we start with the map

$$c\partial\xi\xi e^{-2\phi} \sim \frac{\theta_+}{\lambda^+} \quad (15)$$

and will try to deduce the correspondence between PS and RNS variables by using this isomorphism. Since the Green-Schwarz variable θ^α is known to be related to RNS spin operator by the field redefinition

$$\theta^\alpha \sim e^{\frac{\phi}{2}}\Sigma^\alpha \quad (16)$$

, we write $\theta^+ = e^{\frac{\phi}{2}} \Sigma^+$ where Σ^+ is the component of Σ^α with five pluses (+ + + + +) in the $(\pm)^5$ representation. For our purposes, it is convenient to split 32-component spin operator into two 16-component spin operators Σ_α and $\tilde{\Sigma}_\alpha$ with opposite GSO parities. Then the RNS expression for $\frac{1}{\lambda^+} \equiv (\lambda^+)^{-1}$ which OPE with θ^+ gives L is given by

$$(\lambda^+)^{-1} = ce^{2\chi - \frac{5}{2}\phi} \tilde{\Sigma}^+ \quad (17)$$

where $\tilde{\Sigma}^+$ is the (− − − −) component of the 32-component spin operator (so it has GSO parity opposite to Σ^+). One can easily check that the OPE of $(\lambda^+)^{-1}$ with θ^+ is non-singular, with the zero order term given precisely by L . Next, the λ^+ operator can be read off the OPE

$$(\lambda^+)^{-1}(z) \lambda^+(w) \sim 1 + O(z - w) \quad (18)$$

It is easy to see that

$$\lambda^+ = be^{\frac{5}{2}\phi - 2\chi} \Sigma^+ \quad (19)$$

is precisely the operator satisfying this OPE identity. Note that λ^+ and θ^+ have the same GSO parity. It is now straightforward to identify

$$\lambda^\alpha \sim be^{\frac{5}{2}\phi - 2\chi} \Sigma^\alpha \quad (20)$$

however such an identification is not yet complete for the following reason. On one hand, the expression (20) of λ^α in terms of RNS variables does have some basic properties of pure spinors: it is the dimension zero primary field, it is a commuting spinor (since it is multiplied by the b-ghost which is worldsheet fermion) however its full OPE does not yet satisfy the pure spinor constraint as

$$\begin{aligned} \lambda^\alpha(z) \lambda^\beta(w) \sim & \frac{1}{(z-w)^2} \partial b b e^{5\phi - 4\chi} \gamma_{\alpha\beta}^m \psi_m + \frac{1}{4} \partial b b e^{5\phi - 4\chi} \gamma_{\alpha\beta}^m \partial^2 \psi_m \\ & + \partial b b e^{5\phi - 4\chi} \gamma_{m_1 \dots m_5}^{\alpha\beta} \psi^{m_1} \dots \psi^{m_5} + \dots \end{aligned} \quad (21)$$

so the while the second term of the normally ordered part of this OPE would vanish after substituting into the left hand side of the pure spinor constraint $\lambda \gamma^m \lambda$ (since it would produce the factor proportional to $\sim \text{Tr}(\gamma^m \gamma^{m_1 \dots m_5}) = 0$, the first term would still contribute. In addition, the OPE (21) has a double pole singularity while the OPE of two λ 's in the pure spinor formalism is known to be non-singular [6] The reason is that both the OPE singularity and the violation of the pure spinor constraint are related to BRST non-invariance of the operator (20), while the actual pure spinor must be BRST-invariant.

For this reason, one has to add the correction terms to the r.h.s. of (20) to ensure the BRST-invariance. This can be done by replacing

$$\lambda^\alpha \rightarrow \lambda^\alpha - L\rho^\alpha \quad (22)$$

where $\rho^\alpha = [Q_{brst}, be^{\frac{5}{2}\phi-2\chi}\Sigma^\alpha]$ is the BRST commutator with the right hand side of (20). Since $\{Q_{brst}, L\} = 1$ and $[Q_{brst}, \rho^\alpha] = 0$, the modified λ^α will be BRST-invariant by construction. Evaluating ρ^α and its normally ordered product with L we find the complete RNS representation for the pure spinor variable λ^α to be given by

$$\lambda_\alpha = be^{\frac{5}{2}\phi-2\chi}\Sigma_\alpha + 2e^{\frac{3}{2}\phi-\chi}\gamma_{\alpha\beta}^m\partial X_m\tilde{\Sigma}^\beta - 2ce^{\frac{1}{2}\phi}\Sigma_\alpha\partial\phi - 4ce^{\frac{1}{2}\phi}\partial\Sigma_\alpha \quad (23)$$

It is straightforward to check that this expression for λ^α does satisfy the pure spinor condition (4) (see the Appendix). Note that $\lambda^\alpha = -4\{Q_{brst}, \theta^\alpha\}$ where the factor of -4 is related to our normalization choice in (15). Note that λ^α is annihilated by inverse picture-changing operator $\Gamma^{-1} = 4c\partial\xi e^{-2\phi}$ and therefore cannot be transformed to pictures lower than $\frac{1}{2}$, such as $-\frac{1}{2}$ or $-\frac{3}{2}$. In the next section we will use the RNS expression (23) for λ^α in order to map the BRST charge of pure spinor string theory into RNS BRST charge.

RNS BRST Operator from Pure Spinor BRST Operator

In this section we will use the RNS representation (23) for the pure spinor variable λ^α to construct the exact map relating pure spinor BRST charge and RNS BRST charge. To demonstrate this relation we have to calculate the normally ordered expression of the pure spinor BRST current : $\lambda^\alpha d_\alpha$: in the RNS formalism. The constraint operator

$$d_\alpha = p_\alpha - \frac{1}{2}\theta^\beta\gamma_m^{\alpha\beta}\partial X_m - \frac{1}{8}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha \quad (24)$$

consists of three terms, so we are to calculate the normally ordered OPE's of these terms with λ^α one by one. A useful formula for our calculation is the OPE between two spin operators:

$$\begin{aligned} \Sigma_\alpha(z)\Sigma_\beta(w) &\sim \frac{\gamma_{\alpha\beta}^m\psi_m(\frac{z+w}{2})}{(z-w)^{\frac{3}{4}}} + \frac{1}{6}(z-w)^{\frac{1}{4}}\gamma_{\alpha\beta}^{mnp}\psi_m\psi_n\psi_p(\frac{z+w}{2}) + \dots \\ \Sigma_\alpha(z)\tilde{\Sigma}_\beta(w) &\sim \frac{\delta_{\alpha\beta}}{(z-w)^{\frac{5}{4}}} + \frac{1}{2}(z-w)^{\frac{3}{4}}\partial\psi_m\psi^m(\frac{z+w}{2}) + \dots \end{aligned} \quad (25)$$

where we skipped higher order OPE terms as well as those not contributing to the normally ordered expression for : $\lambda^\alpha d_\alpha$:. Note that, as the ordered RNS expressions for three terms

(24) of d_α contain zero, one and three gamma-matrices respectively (see below), only the terms with one or three gamma-matrices in the $\Sigma\Sigma$ or $\tilde{\Sigma}\tilde{\Sigma}$ operator products and only the terms proportional to $\delta_{\alpha\beta}$ in the $\Sigma\tilde{\Sigma}$ OPE contribute to the BRST current. All other OPE terms (i.e. those with the number of antisymmetrized gamma-matrices other than 0,1 or 3) are irrelevant to us since their contributions to $:\lambda^\alpha d_\alpha:$ produce terms proportional to vanishing traces of antisymmetrized gamma-matrices.

We start with the p_α term of d_α . Since p_α is canonically conjugate to θ^β :

$$p_\alpha(z)\theta^\beta(w) \sim \frac{\delta_\alpha^\beta}{z-w} \quad (26)$$

the RNS representation for p_α is easily deduced to be

$$p_\alpha = e^{-\frac{1}{2}\phi}\tilde{\Sigma}_\alpha \quad (27)$$

i.e. it is simply the space-time supercurrent at picture $-\frac{1}{2}$. Then the normally ordered OPE's of p_α with the first two terms of λ^α are easily evaluated to give

$$\begin{aligned} p_\alpha(z)be^{\frac{5}{2}\phi-2\chi}\Sigma^\alpha(w) &= (z-w)^0be^{2\phi-2\chi}\left(\frac{z+w}{2}\right) + O(z-w) \\ p_\alpha(z)2e^{\frac{3}{2}\phi-\chi}\gamma_{\alpha\beta}^m\tilde{\Sigma}(w) &= (z-w)^0e^{\phi-\chi}\psi^m\partial X_m\left(\frac{z+w}{2}\right) + O(z-w) \end{aligned} \quad (28)$$

so the result is given by easily recognizable (up to normalization factors) ghost and matter supercurrent terms of j_{brst} in the RNS formalism. The OPE of p_α with the remaining part of λ^α , namely, $ce^{\frac{1}{2}\phi}\Sigma_\alpha(2\partial\sigma - 2\partial\phi) - 6ce^{\frac{1}{2}\phi}\partial\Sigma_\alpha$ is a bit more tedious but straightforward to calculate producing terms with the structure $\sim cG_{(2)}(\psi, \sigma, \phi, \chi)$ with G_2 being an operator of conformal dimension two, consisting of ψ, ϕ, χ and σ worldsheet fields, giving a hint on the relevance of this contribution to the $cT + b\partial cc$ part of Q_{brst} in the RNS description. Performing the calculation and collecting all the terms together we obtain the contribution of $:p_\alpha\lambda^\alpha:$ to j_{brst} to be given by

$$\begin{aligned} :p_\alpha\lambda^\alpha: &= \delta_\alpha^\alpha\{\gamma^2b + \gamma\psi^m\partial X_m \\ &+ c\left(\frac{11}{4}\partial\psi^m\psi_m - \frac{13}{16}(\partial\phi)^2 + \partial^2\phi + \frac{1}{16}(\partial\sigma)^2 - \frac{15}{16}\partial^2\sigma - \frac{3}{4}\partial\phi\partial\sigma\right)\} \end{aligned} \quad (29)$$

The next step is to calculate the contribution stemming from the normally ordered OPE of λ^α with the second term of d_α , given by $-\frac{1}{2}\partial X_m(\gamma^m\theta)_\alpha$. However, an important remark should be made first. Since the RNS expressions (16) for θ_α and (23) for λ^α are both at the ghost picture $\frac{1}{2}$, the straightforward evaluation of their OPE would give an operator

at picture 1. This is not quite what we are looking for since all the terms of j_{brst} are at picture zero. Since we expect that the OPE of λ^α and $-\frac{1}{2}\partial X_m(\gamma^m\theta)_\alpha$ reproduces only a part of j_{brst} , the resulting operator is generally off-shell, so one cannot picture transform it in a straightforward manner. As for λ^α , although it is on-shell, inverse picture-changing still isn't applicable to it, as was noted above. For this reason, in order to get a picture zero result for this contribution, instead of taking θ^α in the standard form (16) one has to take it in its equivalent form

$$\theta_\alpha = -4ce^{\chi - \frac{3}{2}\phi}\Sigma_\alpha \quad (30)$$

which is at picture $-\frac{1}{2}$. Although picture-changing transformation isn't well-defined for off-shell variables such as θ^α , the expressions (16) and (30) are equivalent since they both satisfy the same canonical relation with the conjugate momentum p_α . Indeed, since the worldsheet integral of p_α is on-shell, one can transform it to picture $\frac{1}{2}$ obtaining

$$p_\alpha = -\frac{1}{2}e^{\frac{1}{2}\phi}\gamma_{\alpha\beta}^m\Sigma^\beta\partial X_m - \frac{1}{4}be^{\frac{3}{2}\phi - \chi}\tilde{\Sigma}_\alpha \quad (31)$$

. Applying p_α of (31) to θ^β of (30) one easily finds that, while the first term of p_α doesn't contribute to the simple pole of its OPE with θ^β , the second term's OPE with θ produces precisely the simple pole leading to the standard canonical relation. Thus

$$-\frac{1}{2}\theta^\beta\gamma_{\alpha\beta}^m\partial X_m = 2ce^{\chi - \frac{3}{2}\phi}\Sigma^\beta\gamma_{\alpha\beta}^m\partial X_m \quad (32)$$

Evaluating the OPE of this term with λ^α of (23) we obtain

$$2ce^{\chi - \frac{3}{2}\phi}\Sigma^\beta\gamma_{\alpha\beta}^m\partial X_m(z)be^{\frac{5}{2}\phi - 2\chi}\Sigma^\alpha(w) = (z-w)^0\delta_\alpha^\alpha e^{\phi - \chi}\psi^m\partial X_m\left(\frac{z+w}{2}\right) + O(z-w) \quad (33)$$

for the product of (32) with the first term of λ^α

$$\begin{aligned} & 2ce^{\chi - \frac{3}{2}\phi}\Sigma^\beta\gamma_{\alpha\beta}^m\partial X_m(z)2e^{\frac{3}{2}\phi - \chi}\tilde{\Sigma}^\gamma\gamma_{\alpha\gamma}^m\partial X_m(w) \\ &= \delta_\alpha^\alpha c\{2\partial X_m\partial X^m - 8\partial\psi_m\psi^m - 2\partial^2\sigma - 2(\partial\sigma)^2 \\ & - 18(\partial\phi)^2 - 8(\partial\chi)^2 + 24\partial\chi\partial\phi - 8\partial\chi\partial\sigma + 18\partial\phi\partial\sigma\}\left(\frac{z+w}{2}\right) + O(z-w) \end{aligned} \quad (34)$$

for the product of (32) with the second term of λ^α

$$\begin{aligned} & 2ce^{\chi - \frac{3}{2}\phi}\Sigma^\beta\gamma_{\alpha\beta}^m\partial X_m(z) - 2ce^{\frac{1}{2}\phi}\Sigma_\alpha\partial\phi - 4ce^{\frac{1}{2}\phi}\partial\Sigma_\alpha(w) \\ &= -\delta_\alpha^\alpha \frac{7}{2}dcce^{\chi - \phi}\psi_m\partial X^m\left(\frac{z+w}{2}\right) + O(z-w) \end{aligned} \quad (35)$$

for the product of (32) with the third term of λ^α

Note the appearance of an extra $\gamma\psi_m\partial X^m$ term on the r.h.s. of the OPE (33) that ensures the correct normalisation of the matter supercurrent term with respect to the ghost supercurrent term in j_{brst} . The final contribution to j_{brst} comes from the OPE of λ^α and $-\frac{1}{8}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha = \frac{1}{8}\theta^\beta\theta^\rho\partial\theta^\lambda\gamma_{\beta\lambda}^m(\gamma_m)_{\alpha\rho}$. To ensure that the contribution of this OPE to j_{brst} is at picture zero, it is convenient to take θ_β and θ_ρ at the picture $-\frac{1}{2}$ representation (30) while keeping $\partial\theta_\lambda$ at the picture $\frac{1}{2}$ version (16). Using the OPE (25) one easily finds

$$:\theta_\beta\theta_\rho:(z) = \frac{8}{3}\gamma_{\beta\rho}^{mnp}\partial cce^{2\chi-3\phi}\psi_m\psi_n\psi_p(z) \quad (36)$$

Calculating the operator product of (36) with $\partial\theta^\lambda = \partial(e^{\frac{1}{2}\phi}\Sigma^\lambda)$ using (25) gives

$$-\frac{1}{8}:(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha:(z) = -32\partial cce^{2\phi-\frac{5}{2}\phi}(\partial\Sigma_\alpha - \frac{19}{6}\partial\phi\Sigma_\alpha)(z) \quad (37)$$

The calculation of the OPE of (37) with the first term of λ^α (23) gives

$$\begin{aligned} & -32\partial cce^{2\phi-\frac{5}{2}\phi}(\partial\Sigma_\alpha - \frac{19}{6}\partial\phi\Sigma_\alpha)(z)be^{5\phi-2\chi}\Sigma^\alpha(w) \\ &= (z-w)^0\delta_\alpha^\alpha c\left\{\frac{29}{4}\partial\psi_m\psi^m - \frac{179}{16}(\partial\phi)^2 - 13\partial^2\phi + 10(\partial\chi)^2 + 14\partial^2\chi\right. \\ & \left. - \frac{169}{16}(\partial\sigma)^2 + \frac{167}{16}\partial^2\sigma + 24\partial\chi\partial\phi + \frac{59}{4}\partial\phi\partial\sigma - 16\partial\chi\partial\sigma\right\}\left(\frac{z+w}{2}\right) + O(z-w) \end{aligned} \quad (38)$$

The OPE of (37) with the second term of λ^α gives

$$-32\partial cce^{2\phi-\frac{5}{2}\phi}(\partial\Sigma_\alpha - \frac{19}{6}\partial\phi\Sigma_\alpha)(z)2e^{\frac{3}{2}\phi-\chi}\gamma_m^{\alpha\beta}\partial X^m\tilde{\Sigma}^\beta = -\frac{25}{2}\partial cce^{\chi-\phi}\psi_m\partial X^m(z) + O(z-w) \quad (39)$$

Finally, the OPE of (37) with the third term of λ^α produces

$$\begin{aligned} & -32\partial cce^{2\phi-\frac{5}{2}\phi}(\partial\Sigma_\alpha - \frac{19}{6}\partial\phi\Sigma_\alpha)(z)ce^{\frac{1}{2}\phi}(-4\partial\Sigma^\alpha - 2\Sigma^\alpha\partial\phi)(w) \\ &= 32\partial^2 c\partial cce^{2\chi-2\phi}\left(\frac{z+w}{2}\right) + O(z-w) \end{aligned} \quad (40)$$

Collecting together all the terms in (29) - (40) we find the overall normally ordered product of d_α and λ^α to be given by:

$$\begin{aligned} & \frac{1}{16}:\lambda^\alpha d_\alpha: = 2\gamma\psi_m\partial X^m + \gamma^2b + c\{2\partial X_m\partial X^m + 2\partial\psi_m\psi^m \\ & \quad - 18(\partial\chi)^2 + 14\partial^2\chi - 30(\partial\phi)^2 - 12\partial^2\phi - \frac{25}{2}(\partial\sigma)^2 + \frac{15}{2}\partial^2\sigma \\ & \quad + 48\partial\chi\partial\phi - 24\partial\chi\partial\sigma + 32\partial\phi\partial\sigma\} - 16\partial cce^{\chi-\phi}\psi_m\partial X^m + 32\partial^2 c\partial cce^{2\chi-2\phi} \end{aligned} \quad (41)$$

where the factor of $\frac{1}{16}$ in front of the pure spinor BRST current is to absorb the factor of $\delta_\alpha^\alpha = 16$ always appearing on the right hand side of the operator products (29) - (40). Although the RNS expression (41) for the pure spinor BRST current looks tedious, it is straightforward to check that, up to an overall numerical factor and BRST trivial terms, it is equivalent to the BRST current in RNS formalism. Indeed, using the bosonized expression for RNS BRST current:

$$\begin{aligned} j_{brst}^{RNS} &= cT + b\partial cc - \frac{1}{2}\gamma\psi_m\partial X^m - \frac{1}{4}b\gamma^2 \\ &= c\left\{-\frac{1}{2}\partial X_m\partial X^m - \frac{1}{2}\partial\psi_m\psi^m - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi \right. \\ &\quad \left. + \frac{9}{8}(\partial\sigma)^2 + \frac{1}{8}\partial^2\sigma\right\} - \frac{1}{2}e^{\phi-\chi}\psi_m\partial X^m - \frac{1}{4}be^{2\phi-2\chi} \end{aligned} \quad (42)$$

and the commutator:

$$\begin{aligned} [Q_{brst}^{RNS}, \partial cce^{2\chi-2\phi}\partial\chi] &= \partial^2 c\partial cce^{2\chi-2\phi} - \frac{1}{2}\partial cce^{\chi-\phi}\psi_m\partial X^m \\ -\frac{1}{4}c\{2\partial^2\phi - 2\partial^2\chi - \partial^2\sigma + 4(\partial\phi)^2 + 2(\partial\chi)^2 + (\partial\sigma)^2 - 6\partial\chi\partial\phi + 3\partial\chi\partial\sigma - 4\partial\phi\partial\sigma\} \end{aligned} \quad (43)$$

one easily finds

$$\frac{1}{16}j_{brst}^{purespinor} = -4j_{brst}^{RNS} + 32[Q_{brst}^{RNS}, \partial cce^{2\chi-2\phi}\partial\chi] \quad (44)$$

This concludes the calculation identifying the BRST charges in RNS and pure spinor approaches. Note that a shift of a BRST charge by any BRST trivial term (that particularly occurs in (44)):

$$Q_{brst} \rightarrow Q_{brst} + [Q_{brst}, R] \quad (45)$$

where R is some operator, is equivalent to the similarity transformation

$$Q_{brst} \rightarrow e^{-R}Q_{brst}e^R \quad (46)$$

considered in [7]. In our case,

$$R = 32 \oint \frac{dz}{2i\pi} \partial cce^{2\chi-2\phi}\partial\chi(z) \quad (47)$$

Note that the R -operator isn't generally required to be in the "small operator algebra" and, as a matter of fact, both the R -operator (47) and the R -operator used in the similarity transformation in [7] are outside the small algebra: the R -operator (47) contains the factor

of $e^{2\chi}\partial\chi = \frac{1}{2}\partial^2\xi\xi$, while the R-operator used by Berkovits explicitly depends on $\frac{1}{\lambda_+}$ which, when translated into RNS language, isn't in the small algebra as well.

Discussion. Vertex Operators with Non-trivial Pure Spinor Couplings

In this letter we have proposed an exact map expressing the pure spinor variable λ^α in terms of BRST invariant RNS operator of conformal dimension zero, satisfying pure spinor constraint. The map is based on identifying the $\frac{\theta_+}{\lambda_+}$ operator in the pure spinor formalism and the L -operator in the RNS description satisfying $\{Q_{brst}, L\} = 1$. This map particularly leads to the identification (23) of pure spinor and RNS BRST operators, up to similarity transformation (or BRST-trivial terms). The non-triviality of vertex operators in pure spinor approach requires the introduction of “small” and “large” operator algebra in pure spinor approach, similarly to the classification existing in RNS approach. However, classifying the full operator algebra in terms of “large” and “small” appears somewhat ambiguous in the pure spinor formalism, compared to RNS formalism, where such a classification is clear and is based on the bosonization relations for superconformal ghosts. The small operator algebra of the pure spinor formalism should particularly exclude operators inverse to pure spinor components, such as $\frac{1}{\lambda_+}$, but such a constraint appears too relaxed and also somewhat artificial since, unlike RNS variable ξ , which can only be expressed as a generalized step function of superconformal β -ghost: $\xi = \Theta(\beta)$ pure spinor operator $\frac{1}{\lambda_+}$ is the function of a variable manifestly present in the theory. This particularly leads to the pure spinor BRST cohomology containing operators which physical meaning is unclear. In particular any function $F(\lambda)$ is an invariant operator in pure spinor formalism. If $F(\lambda)$ is polynomial, e.g. $F(\lambda) \sim \lambda_{\alpha_1} \dots \lambda_{\alpha_n}$, it can be represented as a BRST commutator $F(\lambda) = \{Q_{brst}, \theta_{\alpha_1} \lambda_{\alpha_2} \dots \lambda_{\alpha_n}\}$, i.e. it is BRST exact. If, however, $F(\lambda)$ isn't a polynomial function (e.g. $F(\lambda) \sim \log(\lambda)$) then the only way to represent it as a BRST commutator seems to be $F(\lambda) = \{Q_{brst}, \frac{\theta_+}{\lambda_+} F(\lambda)\}$, but this doesn't make an operator unphysical, due to the small/large algebra classification. Apparently not all these operators, while formally in the cohomology, are of physical significance. For this reason, one needs to find the way to eliminate these clearly excessive states, which apparently requires better understanding of how operator formalism works in the pure spinor approach.

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Appendix

In this short appendix we demonstrate that the BRST-invariant RNS expression (23) for λ^α satisfies the pure spinor condition (4). Since $\lambda^\alpha \sim \{Q_{brst}, \theta^\alpha\}$, it is sufficient to show that the operator $N^m = (\theta\gamma^m\lambda)$ is BRST-invariant. Taking $\theta^\alpha = e^{\frac{1}{2}\phi}\Sigma^\alpha$ according to (16) and evaluating its OPE with λ^α of (23) using (25) it is straightforward to calculate

$$\begin{aligned}
-\frac{1}{4} : \theta\gamma^m\lambda : &= -\frac{1}{4}\gamma_{\alpha\beta}^m : e^{\frac{1}{2}\phi}\Sigma^\alpha(z)[be^{\frac{5}{2}\phi-2\chi}\Sigma_\beta \\
&\quad + 2e^{\frac{3}{2}\phi-\chi}\gamma_{\beta\lambda}^m\partial X_m\tilde{\Sigma}^\lambda - 2ce^{\frac{1}{2}\phi}\Sigma_\beta\partial\phi - 4ce^{\frac{1}{2}\phi}\partial\Sigma_\beta](z) : \\
&= -\frac{1}{2}e^\phi(\psi_n\partial X^n)\partial X^m + \frac{1}{2}ce^\phi[\frac{1}{2}\partial^2\psi^m + \partial\psi^m(\partial\phi - \partial\chi) + \frac{1}{2}(\partial^2\phi - \partial^2\chi \\
&\quad + (\partial\phi - \partial\chi)^2)\psi^m] - \frac{1}{4}\partial^2ce^\phi\psi^m + \frac{1}{2}c\partial(e^\phi\partial\chi\psi^m) \\
&\quad - \frac{1}{8}e^{2\phi-\chi}[2\partial^2\phi + 2\partial^2\chi - \partial^2\sigma + (2\partial\phi - 2\partial\chi - \partial\sigma)^2]\partial X^m \quad (48) \\
&\quad - \frac{1}{4}e^{2\phi-\chi}[(\psi_n\partial X^n)\partial\phi + \partial(\psi_n\partial X^n)\psi^m - \partial\chi\partial^2X^m - \partial\chi(\partial\phi - \partial\chi)\partial X^m \\
&\quad + \frac{1}{2}\partial^3X^m + (\partial\phi - \partial\chi)\partial^2X^m + \frac{1}{2}(\partial^2\phi - \partial^2\chi + (\partial\phi - \partial\chi)^2)\partial X^m] \\
&\quad - \frac{1}{8}be^{3\phi-2\chi}[(2\partial\phi - 2\partial\chi - \partial\sigma)(2\partial\phi - \partial\chi - \sigma) + 2\partial^2\phi - 2\partial^2\chi - \partial^2\sigma]\psi^m \\
&\quad - \frac{1}{4}[Q_{brst}, e^\phi(\psi^m\partial\phi + \frac{1}{2}\partial\psi^m)]
\end{aligned}$$

Up to the BRST trivial terms, the right hand side of (48) can be recognized as $[Q_{brst}, \xi V_{photon}]$ where $V_{photon} = c\partial X^m + \frac{1}{2}\gamma\psi^m$ is the unintegrated photon vertex operator at zero momentum. For this reason, the RNS expression for $-\frac{1}{4}\theta\gamma^m\lambda$ is given by the unintegrated photon vertex operator at superconformal ghost picture 1 at zero momentum (which of course is BRST-invariant) plus BRST trivial terms. Therefore $-\frac{1}{4}\theta\gamma^m\lambda$ is BRST-invariant and its BRST commutator, given by $\lambda\gamma^m\lambda$, is identically zero. This concludes the proof that λ^α satisfies the pure spinor constraint (4).

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